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# RELATIVISTIC HYDRODYNAMICS

by

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## Introduction

by

Craig Hudson

Today we have Professor Taub from the University of Illinois. He is Research Professor at the Illinois Computing Laboratory. This has nothing to do with the talk but I think it would be nice if he would give us about a five-minute rundown on what the Computing Laboratory does. We are all interested in that sort of thing. The actual talk is going to be on relativistic hydrodynamics. Professor Taub.

## Professor Taub

To begin with a description of the Digital Computer Laboratory at the University of Illinois, I can tell you a little about its history and some of its plans and what it is going to do. The Digital Computer Laboratory started in about 1948. At that time the acquisition of computers was a little different than it is now. The laboratory then was very much interested in the field of computer development, computer use, and development of mathematical methods for using computers efficiently, and felt that these purposes could best be achieved if a high-speed computer were available.

We decided that the best and cheapest way to get a high-speed computer was to build our own, and we started building a machine patterned very much after that being developed by von Neumann and his group at the Institute for Advanced Study. We built two of these machines. One was delivered to Aberdeen and called the Ordvac. The second we kept for ourselves and called the Illiac. There have since been copies of this machine. One is in Sydney, Australia, and it is called the Silliac.

For the past five years or so we have been using the machine, developing methods, and building a library for it. We began to feel pinched in the type of problem we were able to do on it and decided it was time to try for a higher speed machine. We have completed the design and started the construction of a machine which we think will be of the order of one hundred to two hundred times faster than the Illiac. I do not know when it will be finished; I will not say when we think it will be finished. You cannot apply the usual pi-factor to it. This construction program has been launched.

At the same time we have an active group interested very much in numerical methods, in devising schemes for handling problems more effectively. We also have a group interested in various problems of applied mathematics, and the reason they are in the Computer Laboratory is twofold. One is partly accidental. Secondly, many of the problems we are interested in have to do with nonlinear partial differential equations about which nobody knows any general theorem. Therefore, we are building up a stock of examples which we find of interest. In these examples, we include problems of hydrodynamics and problems of relativity. This is where I start talking about relativistic hydrodynamics, so enough about the laboratory.

I thought this morning that I would try to give you what might be called a conceptual tour through relativistic hydrodynamics instead of any detailed examples. I shall just talk about the theory instead of proving any theorem for you or trying to tell you of some of the mathematical difficulties that arise. One of the first problems one has in any theory is the formulation of what you are talking about and what holds you have on achieving some results about the things that you are interested in. The theory of relativity in both its special and general forms is intimately related to hydrodynamics and they both give rise to some conceptual problems.

The first problem we have to talk about is what you mean by hydrodynamics. You mean the theory of describing the motions of an object called the "fluid," and usually you make a large number of idealizations hoping that you still leave in the idealized thing you are dealing with, some essentials of the problem that you are really interested in.

2 Now the fluid, classically as you know, can be viewed from three levels. The first is the continuum level in which you say that you have a continuous medium. This medium has at each point a velocity field, a pressure field, a density field, and there may be other forces involved, in addition to those represented by the quantities I have mentioned.

This is an abstraction from what the physicists would say is actually going on. There is not any continuous medium. There are only the elementary molecules which make up the fluid, and what you are talking about when you talk about the velocity field and the pressure field, and so on, are various averages of relevant quantities; and, finally, that the kinetic theory has ingrained in it a very important function called the distribution function which tells you the number of particles located at a certain position in space and having velocities between certain limits. If you knew the distribution function and if you knew what laws governed it, you could, from it, build up all the averages you wanted, and you would not have to talk about the continuum, and you would have a more accurate representation of what is going on in the medium. Indeed, one way of handling the motion of a fluid for the case of a rarified gas is to go back to the distribution function and see, under certain assumptions, what difference there is in the behavior of the velocity field and the other quantities as compared with the usual case.

Thus, there is the kinetic theory, the Boltzmann theory which I have just mentioned, for determining the distribution function and therefore determining the average quantities which enter into the continuum theory. But there is another method for dealing with the distribution function which is not the Boltzmann method but the method of statistical mechanics which says, indeed we have these molecules. Instead of talking about the average behavior or the behavior of an average molecule and determining something about the distribution function, you can follow or attempt to follow the history of each of the molecules that you are dealing with. This is essentially the level at which statistical mechanics operates.

Now, of course, it is a rather ambitious program, even for our new computer, to deal with the number of molecules that one thinks of in connection with a gas. Nevertheless this, I think, is a rather promising approach to the problem of fluid dynamics for the reason that you may get a better approximation to the behavior of the fluid. If you took a ridiculously small number of molecules, the order of a thousand or two thousand instead of the $10^{23}$ that you really should have, you may still get a better approximation by dealing with that small number than you would if you went through the chain of reasoning of: one, abstracting from this collection of particles; two, smoothing them out and dealing with the average quantity or continuous functions that occur in hydrodynamic theory; then, in order to squeeze it into a computing machine, which is a digital device, you take finite differences.

Our usual pattern really amounts to this, that we have a finite system. We smooth it out and replace it by quantities of interest, by average quantities. We cannot deal with the average quantities as such with digital computers and, therefore, we go back to another finite system. But this chain is far removed from the original finite system, and, in the process of going through this chain, we have made enough assumptions so that we might really not get results of real interest or, in some problems, not really be able to handle what we want. So much then for this view of hydrodynamics.

The view involving the Boltzmann function, or a distribution function determined from statistical mechanical principles, is the view I want to take in describing what we are going to talk about in relativistic hydrodynamics. There is one conceptual problem that we have to get over here. As you all know, the Boltzmann function, even for a gas at equilibrium, is sort of a Gaussian function. It has tails on it and the tails have arbitrarily high velocities. You also know that, if you talk about things in special relativity, there is a rule which says you cannot have a velocity bigger than the velocity of light. How do you get over this hump?

The answer to that is fairly simple. You get over this hump by talking about the distribution with respect to momentum instead of velocity. In the classical theory, these two only differ by a simple constant. The momentum is the mass times the velocity. In the theory of special relativity, this is no longer a simple constant. The mass, if you will, depends on the velocity of motion. One has the factor

$$\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} ,$$

and as $v$ approaches $c$ one gets an infinity for the mass of the object that one is dealing with. So we can get around that conceptual difficulty.

Then we can say that in the special theory of relativity, where we are dealing with conglomeration of particles, we are dealing with a conglomeration of particles described by a distribution function, $f$, which I will write as a function of $x$ and a variable $\xi$, the momentum per unit mass, $f(x,\xi)$. We can then define a mean velocity, or the components of a mean velocity, by taking the integral of this:

$$U^{\mu} = \int V^{\mu} f\,(x,\xi)\, d\Omega .$$

The $V^{\mu}$ is then the velocity field of each individual particle and is a function of the same variable, $\xi$, and the indices, $\mu$, now run from one to four instead of one to three, because we are talking about things in a four-dimensional space time instead of the conventional space.

We can also define a quantity called the stress energy of the collection of objects and call this the stress-energy tensor:

$$T^{\mu\nu} = \int V^{\mu} V^{\nu} f(x, \xi)\, d\Omega \ .$$

This has ten components in it, and it is a straightforward generalization of what occurs in the usual classical theory where the stress tensor--the quantities involved if $\mu$ and $\nu$ are restricted to the range from one to three--is defined in terms of the averages of the velocities times the distribution function, averaged over all the possible particles. The $\Omega$ that I have introduced here is an invariant volume element on a three-dimensional surface in the space, because all these velocity fields satisfy the condition:

$$-\left(v^1\right)^2 - \left(v^2\right)^2 - \left(v^3\right)^2 + \left(v^4\right)^2 = 1 \ .$$

Our fluid then is characterized by the ten quantities $T^{\mu\nu}$ and the four $U^{\mu}$. Whenever we talk about a fluid in relativistic hydrodynamics, we are talking about something which mathematically can be described by these fourteen quantities.

Let me point out again that if we knew the function, $f$, that we are dealing with, the analog of the Boltzmann distribution function, then we would know everything about this description of the fluid. In lieu of this, we ask, what can be said about $f$? We may write down the analog of the Boltzmann equation which says that the time derivative or the rate of change of $f$ or the change of the number of particles in a given position at a given time and with the velocity in a given range is due to the flow of the particles into the element of volume in phase space, plus the contribution due to those particles that are knocked into it by collisions with other particles. If one had an $f$ that satisfied this equation, then automatically one would get five conservation laws. I shall write them out in a queer notation for you and tell you what it means, and then give names to these conservation laws:

$$T^{a}{}_{,\nu}{}^{\nu} = 0 \qquad (1)$$

$$U^{\mu}{}_{,\mu} = \left(\rho u^{\mu}\right)_{,\mu} \ . \qquad (2)$$

What this means is that the partial of $U^4$ with respect to t plus the partial of $U^3$ with respect to z , plus the partial of $U^2$ with respect to y , plus the partial of $U^1$ with respect to x , is equal to zero. This comma and repeated index is a shorthand notation. All Equation (2) says is that the mass current is conserved. This law, then, is the conservation of mass. The laws given by Equation (1) are four in number, because $\alpha$ takes on the values one, two, three, and four, and they represent the conservation of momentum and the conservation of energy. The special theory of relativity is concerned simply with a statement about the special solution of these equations. The Equations (1) and (2) reduce to the usual equations of five conservation laws in ordinary hydrodynamics if you simply take the convention that the velocity of light becomes infinite. So this is a straightforward generalization of classical theory, and there is only one more thing to he said.

You can either assume from the kinetic theory (or show in the equilibrium state of the statistical mechanics theory) that this stress-energy tensor has a particular form for a simple fluid, where one neglects viscosity and heat conductivity, and that it can be written in the following way:

$$T^{\mu\nu} = \sigma U^{\mu} U^{\nu} - \frac{p}{c^2} g^{\mu\nu}$$

Instead of being a general symmetric tensor with ten components, it really depends on the velocity field given in Equation (2) $\left(g_{\alpha\beta}\right.$ in the special theory are particular constants$\left.\right)$. We will define the quantity, $\rho$:

$$\rho^2 = U^{\mu} g_{\mu\nu} U_{\nu} = U^{\mu} U_{\nu}$$

This $\rho$ is called the rest density of matter. It is the density of the matter as observed by somebody moving along with the matter.

In some of the older works in relativistic hydrodynamics, the five conservation laws get lost in various places, and this is because there is a confusion in the interpretation of what these five laws mean. The law (2) really says that the number of particles that one has in this background picture never gets changed. Molecules never are created or lost. The law (1) or the energy law in it, says that the rest mass of the particles that I am dealing with contributes to the total energy that I have; and that there is another quantity called the specific internal energy of the gas (the amount of work I could get out of the gas if I expanded it adiabatically out to infinity) which also contributes to the energy. In other words, the $\sigma$ is not $\rho$; actually

$$\sigma = \rho \left(1 + \frac{\epsilon}{c^2} + \frac{p}{\rho c^2}\right) ,$$

this $\epsilon$ being the internal energy per unit mass. Actually, the relativistic effects will come in when the quantity $p/(\rho c^2)$ is large. At normal pressures and densities, that is really a small term, and the difference between relativistic hydrodynamics and ordinary hydrodynamics is not appreciable. It is only when this term is large that one will get extreme relativistic effects, and you can see for yourselves what "large" means here. The quantity $p$ is the pressure in ordinary units, $\rho$ is density, and $c$ is the velocity of light. That has to be large. The expression $p/\rho$ is essentially the velocity of sound squared, except for the $\gamma$-factor. It gets modified a little if you are going to take certain things into account, but for our purposes we can take that as the velocity of sound itself. So it is only when the gas is so hot that the velocity of sound is comparable to the velocity of light that one is really going to get anything exciting out of this, compared to the usual, standard theory. Nevertheless, it is fun to play games, and you never know when somebody is going to do something.

Again, let me point out something to you which I am sure a great number of you are familiar with, that in ordinary hydrodynamics, even if one starts out with things very smooth and well-behaved--take a gas in a tube and just push a piston into the tube--no matter how smoothly you push as long as you decrease the volume of the tube, you are going to get into trouble as far as the differential equations are concerned, because there is a nasty thing called a "shock" that is going to develop and will compress the gas. The differential equations no longer hold across that, but something else does, at least in the approximations that we are making where we are neglecting viscous effects and the heat conductivity. The shock is represented by a mathematical discontinuity, and the laws of these differential equations have to be supplemented with additional rules, called the Rankine-Hugoniot equations, which do hold across the shock.

In the classical formulation of these equations, the velocity of the shock front increases indefinitely with the pressure jump. The velocity of the shock front behaves as follows:

$$M^2 = 1 + \frac{\gamma + 1}{2\gamma} (y - 1) ,$$

where $y$ is the ratio of the pressure behind divided by the pressure in front, and $M$ is the Mach number of the normal component of flow relative to the shock. If $y$ becomes very large, the shock-front velocity becomes large and, again, we are in trouble with the fundamentals of the theory of relativity, namely, by means of shocks one could conceivably get velocities bigger than the velocity of light. How can one get out of that bind? Well, it turns out that you can get out of it.

As soon as we go to the continuum picture, we are tempted to forget the background and say that the internal energy can be any function of $p$ and $\rho$. In particular, one might think that a function of the type

$$\epsilon = \frac{1}{\gamma - 1} \frac{p}{\rho}$$

(the function that is used in classical theory) would be an allowable function here for any value of $\gamma$. Well, it turns out that if you used a value of $\gamma$ equal to two or more in that expression and worked out what the corresponding velocity for the moving shock would be, you would indeed get a contradiction--namely, that the velocity of the shock front divided by the velocity of light for very strong shocks would tend toward $\sqrt{\gamma - 1}$. This says that one would be in contradiction with the fundamental principles of relativity if $\gamma$ is larger than two, because then one would have a wave carrying energy and propagating with a velocity greater than the velocity of light.

So you have to go back and ask yourself, is there anything about the picture that we are dealing with which rules out a function of this type from a $\gamma$ bigger than two? The answer is that there is. Just this fundamental background that has been introduced, the collection of particles you are dealing with, and so on, does put a restriction on the possible $\gamma$ and, indeed, $\gamma$ cannot be bigger than 2, and the function $\epsilon$ conforms with relativity theory.

I say, then, that conceptually one can formulate a reasonable theory in special relativistic hydrodynamics. One can get methods for handling all the easily manageable problems in relativistic hydrodynamics, and by easily manageable problems I mean the same type that can be handled in the classical theory. As you probably know, in classical theory when one is dealing with a one-dimensional motion, say, the motion in a tube filled with gas in which a piston is moved on one side, and then you can give a general solution which will tell you what will happen if you pulled the piston out. If the piston is pushed in, it is not that clean.

The corresponding problems can be done in this theory and, indeed, one can even show that shocks will form exactly in the cases where they formed before. There is nothing about this mysterious subject of special relativity which gets you out of the difficulty due to the nonlinearity in the classical theory.

The next question is, what can be said about the general theory of relativity? Let me give you some background information there. There are two classes of problems that one could consider in that theory. Namely, one may want to talk about the motion of the gas in which one wants to neglect the gravitational field of the gas itself but wants to take into account the gravitational field of other objects which influence the motion of the gas. The other type of problem that one might consider is to say that the only thing you have to worry about is the gas, its motion, and its self-gravitational effects. That there is a distinction between these two problems is, I think, clear conceptually. I want to indicate to you how one handles these problems separately, or at least write down the presumed equations holding for these problems in both cases and then see how one really is an inverted form of the other.

The special theory of relativity says of space time that it is a particular four-space in which time and space are joined to a four-dimensional continuum in a particular manner. In this theory, the "distance" intervals are given by

$$ds^2 = dt^2 - \frac{1}{c^2}\left(dx^2 + dy^2 + dz^2\right) .$$

That is, the space time is characterized by a metric in which the interval between two events in the space time is given by the above equation, the events being distinguished here as one event with coordinates $x$, $y$, $z$, and $t$, and another event having the coordinates, $x + dx$, $y + dy$, $z + dz$, $t + dt$.

The general theory of relativity is general in the sense that it says that the space time around us is not given by anything that specific but is given in terms of ten gravitational potentials in terms of which the "distance" is written this way:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu ,$$

with summation on the indices $\mu$, $\nu$. There are ten quantities $g_{\mu\nu}$, analogs of the gravitational potential, which in the space time of special relativity take on values so that $g_{11}$, $g_{22}$, $g_{33}$, or $g_{44}$ are the only nonvanishing $g$'s in this expression, and they satisfy

$$g_{11} = g_{22} = g_{33} = -\frac{1}{c^2}\ , \quad g_{44} = 1 .$$

Now, Einstein went further and not only generalized space time but actually gave a rule or a set of rules for calculating what the $g$'s were. He stated that the nature of the space time that we are dealing with is determined (that is, these ten quantities the g's, are determined) by the amount and nature of the matter present.

In other words, there are a set of ten second-order differential equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -kc^2 T^*_{\mu\nu} . \tag{3}$$

In this, $k = 8\pi G/c^2$, G is the Newtonian constant of gravitation, and these are the ten differential equations for determining the g's. The matter present is described by the ten numbers $T_{\mu\nu}$. These ten differential equations are not very simple. They are second-order partial differential equations, nonlinear, and all ten quantities enter in almost all ten equations. They are a highly coupled system of ten nonlinear partial differential equations.

Now, if I want to know the motion of the gas in a space time where there is so much matter present that I can forget about the gravitational effects of the gas itself, then what I do is to determine the g's by solving Equation (3) and generalizing the five conservation Equations (1) and (2) slightly. What happens to the Equation (2) is that it becomes the equation,

$$\left(\rho u^{\mu}\right); \mu = \frac{1}{\sqrt{-g}} \left(\sqrt{-g}\, \rho u^{\mu}\right), \mu = 0 , \tag{2'}$$

$$g = \det \left\| g_{\mu\nu} \right\| .$$

This equation reduces to Equation (2) when the g's become these constants. The g's are assumed known from Equation (3) in which $T^*_{\mu\nu}$ do <u>not</u> refer to the gas.

The next set of conservation laws, Equation (1), become a little more involved:

$$T^{\mu\nu}; \nu = \frac{1}{\sqrt{-g}} \left(\sqrt{-g}\; T^{\mu\nu}\right), \nu + T^{\rho\sigma} \left\{ {}_{\rho}{}^{\mu}{}_{\sigma} \right\} . \tag{1'}$$

The second term in this equation takes into account the gravitational forces. This is the straight generalization of Equation (1) involving the g's, which reduces to (1) when the g's are constant. These are the conservation laws and if I determine the g from Equation (3), then the equations describing the hydrodynamics are Equations (1') and (2'), and the formulation of the problem is complete. But the solution is far from complete, because we do not know the $g_{\mu\nu}$.

Now, what happens when the only matter present is the gas that we want to describe, and we do not want to neglect its own gravitational effect on its own motion? What happens, then, is that $T^{*}_{\mu\nu}$ in Equation (3) is the $T^{\mu\nu}$ of Equation (1'). Equations (1') are not additional assumptions but are the consequence of Equations (3). In other words, if I say that the fluid is the thing which creates the gravitational field, and I want the motion of the fluid in its own gravitational field, I must remove the star in Equation (3), and now I have a set of equations which are really eleven in number, the ten given by Equation (3), Equation (2'), and the conservation Equations (1') are automatically satisfied as a consequence of Equation (3).

Now, what can be said about this business? Let me point out to you that we have rather a large number of unknowns. We have the quantity $\rho$, (one), we have the velocity field which has three independent components because I have normalized it (that is four so far), and we have the pressure (which is five). The $\epsilon$ is presumed known as soon as I know the pressure and the density. I change the kind of gas I am talking about by changing the $\epsilon$. For example, if I take $\epsilon$ identically zero, then my gas is what is usually referred to as an incompressible fluid. If I take $\epsilon$ to be

$$\frac{1}{\gamma - 1} \frac{p}{\rho},$$

where $\gamma$ is 5/3, then it is a monatomic gas. So I change the kind of gas by talking about $\epsilon$, but it is not another unknown in this unknown count. In addition to these five unknows there are 10 g's and that means fifteen unknowns. Our situation is that in Equations (3) we have differential equations for a set of quantities, the $g_{\mu\nu}$, where we do not even know the right-hand side; and the problem of solving the equations of motion of a fluid, in this case, is the problem of determining the solutions of a set of differential equations of which you know neither the left- nor the right-hand side. The amazing thing is that you can do something about it. You can, indeed, get solutions of this system of equations.

The method of proceeding takes advantage of the fact that the coordinates used above are arbitrary. They are at the disposal of the problem solver and he can pick them at this convenience. One very nice way of picking them is to choose them so that the observer or the coordinate system that one is using moves along with the fluid. In relativity these coordinates are called comoving coordinates. For those of you who have worked with hydrodynamics, let me point out that this is a fancy name for something that a lot of people have been using since Lagrange. In fact, the Lagrange coordinate systems are the so-called comoving ones. You see, the Lagrange coordinates for a fluid label elements of the fluid by their initial position, and from then on the labels of the particles never change. The Eulerian coordinates in classical hydrodynamics are a set of coordinates which are fixed in space and the world lines of the particles (the history of the particles) when drawn on an $x$, $t$ diagram are given by curves,

$$x = x(t)$$

in which the $x$ changes with $t$. If, however, you use Lagrange coordinates where you assign a given label to a certain element of the fluid, that label never changes.

Now, if one introduces comoving coordinates, the analogs of Lagrange coordinates (and you can show that you can introduce them if you make further assumptions about the motions, so that the next assumption I have to make is that the motion is isentropic and irrotational), then you can show that the right-hand sides of these equations are expressible in terms of the same functions as enter the left-hand side, and now one has a determinate set of equations. One can get explicit solutions in some cases. The solutions are going to depend on the nature of $\epsilon$ as a function of $p$ and $\rho$. If you change the kind of gas (the function $\epsilon$) you are going to be able to get solutions, or not be able to.

One can then get a little more ambitious and ask, can one have a scheme for handling these equations in general? The answer is that one is really asking for too much. One could not even handle that problem in the case of classical hydrodynamics. Nobody has the scheme for handling the general flow of a fluid, neglecting its gravitational field, neglecting the fact that the velocity of light is not infinite. However, one can hope to get an approximation scheme, namely, we can say, let us make expansions of the quantities, $g_{\mu\nu}$, which are now the only things entering in these equations as a power series in $k$:

$$g_{\mu\nu} = g_{o\mu\nu} + kg_{1\mu\nu} + k^2 g_{2\mu\nu} + \ldots$$

What this amounts to saying is that you are somehow going to turn on the gravitational field gradually and see what effect this has.

Now, let me call your attention to something that was done some time ago by one of my colleagues at Illinois, McVittie, where he observed the following. There is a $k$ showing up explicitly on the right-hand side of Equation (3). Therefore, the first-order term of the left-hand side is proportional to the zero-order term on the right-hand side. Zero order means classical solutions, or special-relativity solutions, so that the corrections due to the gravitational field will be determined if you know the classical solution. In other words, if at one stage, $g_{n\mu\nu}$ is known, Equations (3) give equations for $g_{n+1\mu\nu}$. This is an obvious remark.

On the other hand, McVittie said, if I know the first-order term of the left side of Equation (3), then I can get the zero-order term of the right side of Equation (3). In other words, I can solve a classical problem if I take a arbitrary $g$ such that the form of the right-hand side of Equation (3) is the form of the $T_{\mu\nu}$ for a fluid. He devised a scheme for determing or using, if you will, the general theory of relativity to go backward to solve classical problems, and he has written a number of papers on this subject.

By this time the fluid problem and the gravitational problem are very much interwolven, due to our use of a Lagrange (i.e., comoving) coordinate system. We postulate, as seems reasonable, in making these expansions, that the $g_{o\mu\nu}$ are the $g_{o\mu\nu}$ of the special theory, but, remember, the special theory in Lagrange coordinates. Therefore, the $g_{o\mu\nu}$ are not given by constants but by their appropriate values which involve a solution of a special relativistic problem in hydrodynamics. One can use these to get a set of linear equations for the $g_1$ and another set of linear equations for the $g_2$, and so on.

Now, I want to call another problem to your attention. I have emphasized about three times already that in the classical theory, and in the special relativity theory, we get into a mathematical difficulty represented by the creation of shock. Let us not argue whether shocks really exist or not. The notion of a shock is a mathematical idealization of an abrupt zone of transition. What is going to happen if we start out with a problem in this general theory in which the corresponding problem in the special theory would have a shock? If I can use this approximation scheme, this means that the $g_o$ goes haywire. It is no longer a continuous function; there is a discontinuity. This means that the differential equation describing the $g_1$ goes haywire. Since the differential equation describing the $g_1$ is a linear one, there is nothing it can do to save itself from that. In other words, we must be prepared to admit that the Einstein field equations, as they are written down here, must be supplemented by the same analogies of the Rankine-Hugoniot equation. In other words, the existance of shock in the hydrodynamic fluid must imply singular surfaces in our space time, across which peculiar things happen to the gravitational field. This is another version of the principle of equivalence.

Peculiar things happen to mass distributions in classical theory. Because of the principle of equivalence between inertial and gravitational mass, peculiar things can happen to the gravitational field or to the acceleration field that you would have to accept in order to straighten things out. Some of my more mathematically inclined colleagues will talk about existence theorems for this set of differential equations subject to continuity conditions of the order of $C^2$ if the functions are continuous and have two continuous derivatives, and so on. This is all very fine indeed, but the fact of the matter is that all you can expect out of this is a theory which should be continuous to the order $C^1$. This means that the differential equations themselves do not even make sense now, because if I have shocks then I cannot form second derivatives, for I know if I replace the shocks by a continuous transition that the second derivatives are going to be very large.

So, if I want to keep the consistent idealization that we have talked about all along, I have to generalize these equations. Let me indicate how you can do that. It will, indeed, be a rough indication. As some of you know, in all of mechanics, you have at your disposal two alternative descriptions of a physical system. One can talk about it in terms of differential equations and initial conditions. One can think about the Newtonian motion of a single particle in a potential field. You can describe this by saying that you have laws which say that the mass times the acceleration is equal to the force acting, that is, the gradient of the field. And if you specify the initial position and initial velocity you determine the motion of the system.

There is another equivalent form, absolutely equivalent, which is philosophically quite different. Namely, you can say, there exists a variational principle such that if you take the integral of the kinetic energy minus the potential energy with respect to time, from $t_1$ to $t_2$, then that motion which we will obtain will be that for which the integral,

$$\int_{t_1}^{t_2} (T - V)\, dt \ ,$$

is a minimum.

Now, you see, you are describing conditions at the end points of the motion. You are not starting out with an initial-value problem but with an end-point problem, but we know that this statement is completely equivalent to the classical one-body problem. This love of people for variational principles is something that has continued ever since d'Alembert found the first one. Indeed, one can ask, is there a variational principle which will give you the equations we have discussed above if you vary the right quantity? The answer is yes. I will just indicate the integral involved:

$$\mathcal{J} = \int \sqrt{-g}\left[R - 2k\rho\left(c^2 + H^o + \frac{1}{2}\,\mu\, g_{\mu\nu}\, U^\mu U^\nu\right)\right] d^4x \ .$$

If you vary the proper quantities in this integral you will get these Equations (1) and (3) out of it.

Now, let me point out this. This variational principle makes sense even if you have a surface of discontinuity, because you have a volume integral to carry out. You can break up this volume integral by integrating on up to the surface of discontinuity on one side and from the other surface up. Therefore, the interal I still makes sense when the equations break down. This is the standard method of generalizing. One gets two equivalent (for some cases) formulations. One chooses that one which makes sense when the first one breaks. Furthermore, if you vary the surface of discontinuity here, you get the Rankine-Hugoniot equation. So the complete theory, which makes sense and allows for these singular hypersurfaces, can be taken care of if you go to the variational principle.

I think I had better stop now instead of talking about generalities, because the only recourse I have is to back up some of my general statements with some calculations, and I think we both had better avoid that.

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Questions from the audience could not be transcribed from the tape recording, but some of Dr. Taub's answers are given below.

1. There are quite a few details that can be mentioned about the thing. The most important thing comes about if this $p/(\rho c^2)$ is indeed large, as I said, then the gas behaves like a combination of photons instead of material particles. This is what you would expect would happen there. The actual difficulty in really giving an honest answer to your question is this. The number of general solutions one has in either of the two cases is so small that the samples from which you can make striking discrepancies are very small indeed. I think that there are many peculiarities. I think there are ways of dealing with particular systems that will illustrate, if you will, reasonable experiments. Although this whole theory, at least the special relativity part of it, has now been known for 50 years, or should have been known for 50 years, not much attention has been paid to this. It was felt that the conceptual structure of the theory was known and there was not much interest in looking at particular examples. I disagree with the latter part, mainly because it seems that one way out of some of the difficulties that are facing modern physical theories is to call upon this mysterious thing known as nonlinearity. I think it is silly to call on it unless you have some examples of what it can do for you. Therefore, I think, there will be more and more interest coming back into these problems. Indeed, a survey of recent literature would substantiate that. There are more papers on relativity, I think, in the last five years than there have been for the preceeding twenty.

2. The integral $\mathcal{J}$ has three terms in it: The first, the scalar curvature of the space time; the second, the generalization of the velocity and the kinetic energy and the third, the Helmholtz free energy which plays the role of the potential.

Actually, the variational principle involving $\mathcal{J}$ was discovered by an evolutionary process. More than ten years ago, I gave a variational principle for a classical fluid. Then, more recently, I gave a variational principle which gave the equations of the theory of relativity and the fluid out of one principle. Finally, I showed that if there are discontinuous surfaces present, you get the well-known equations across them out of the same variational principle. So it grows, naturally. It starts out with the hydrodynamics where the Helmholtz free energy plays the role of a potential.

3. It does depend on the assumption that you can expand in powers of k. Let me say this. If you had any solution of this equation which was obtained by any method whatsoever, it would obviously be a function of k. If the solution were well-behaved, it would have to be a well-behaved function of k; therefore it would have to have this expansion. The only out about this would be that in the case k equals zero, this is not the special relativity solution. This is an added assumption to the theory, that we will start out with k equals zero and have the special relativity solution, that it has an expansion, and that it is analytic in k.

4. I do not know. Various people give various models, and so on. There are really some very fundamental questions involved in this interpretation. For example, as you know, matter in the general theory of relativity is presented in one of two ways, either as a stress-energy tensor, as I have done it up here, or as a singularity in the field. There is no definition which makes any sense of an essential singularity. In other words, you cannot really tell whether matter is present or not, as the present theory stands. This is a horrible confession for someone who believes in the general theory of relativity, but this is true. This means that all these glib and blithe statements made in quite a few text books on general relativity have to be re-examined in the light of the necessities of this physical problem. In other words, what can you say about singularities? What can you say about solutions? What kind of coordinate transformations are you going to allow? The difficulty in realizing what a singularity means is that if you allow certain types of coordinate transformation, then you can either get rid of or introduce singularities. That is clear enough, I am sure, because enough of you have been exposed to tensor analysis to remember that in the new coordinate system, the g's as functions of the new coordinates are related to those in the old coordinate system by equations like this:

$$g_{\mu\nu}{}^{*}\left(x^{*}\right) = g_{\sigma\tau}\left(x\left(x^{*}(1)\right)\right) \frac{\partial x^{\sigma}}{\partial x^{\mu *}} \frac{\partial x^{\tau}}{\partial x^{\nu *}}$$

If either $x(x^*)$ or $\partial x^\sigma / \partial x^{\mu *}$ is singular or misbehaved, then this $g_{\mu\nu}{}^*(x^*)$ will be misbehaved. If all you know is that $g_{\sigma\tau}(x)$ is misbehaved, you can ask yourself, can I get rid of it by introducing a singular transformation? To date, singular transformations have been ruled out and what I am arguing is that you have violated the principles of what you really want because, if you go back again to the space-time pictures, the world lines (say, in the Eulerian coordinate system) are not smooth curves if there is a shock in the problem. What happens is that the world line develops a kink. There has been a discontinuous change in the velocity, and the shock front produced it. If I want to go from these coordinates to a set of Lagrange coordinates, where there is no kink showing, then the coordinate transformation must introduce that kink. Until we understand more about the mathematical theory of these more general coordinate-system transformations, or translate these into some experiments that we can look at in which there are shocks, it is a little hard to say what is going to happen.

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